United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Hamilton paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. A regular polygon P has four more sides than another regular polygon Q, and their interior angles differ by 1°. How many sides does P have?

SOLUTION

Solution 1

Say P has n sides. Then Q has $(n-4)$ sides.

Given that their interior angles differ by 1°, their exterior angles also differ by 1°.

Using p to represent the exterior angle of P and q to represent the exterior angle of Q, we have

$$p = \frac{360}{n} \quad \text{and} \quad q = \frac{360}{n-4}.$$

Since P has more sides than Q, $p < q$, so $p + 1 = q$. Hence

$$\frac{360}{n} + 1 = \frac{360}{n-4}.$$

Multiplying through by $n(n-4)$ gives

$$360(n-4) + n(n-4) = 360n,$$

which simplifies to

$$n^2 - 4n - 1440 = 0.$$

This factorises to give

$$(n+36)(n-40) = 0,$$

which (since n cannot be negative) leads to $n = 40$. Hence P has 40 sides (and Q has 36).

Solution 2

Defining n as before, we can work directly with the interior angles of P and Q (noting that the interior angle of P is larger than the interior angle of Q:

$$\begin{aligned} \frac{180(n-2)}{n} &= \frac{180(n-6)}{n-4} + 1 \\ 180(n-2)(n-4) &= 180n(n-6) + n(n-4) \\ 180n^2 - 1080n + 1440 &= 180n^2 - 1080n + n^2 - 4n, \end{aligned}$$

which leads to the same quadratic equation as in solution 1.

REMARK

The correct factorisation of the quadratic is not immediately obvious. One route to spotting the factors required is to consider how we reached 1440 in the first place: $1440 = 360 \times 4 = 36 \times 40$, which leads to the factorisation straight away.

2. Hudson labels each of the four vertices of a triangular pyramid with a different integer chosen from 1 to 15. For each of the four triangular faces, he then calculates the mean of the three numbers at the vertices of the face.

Given that the means calculated by Hudson are all integers, how many different sets of four numbers could he have chosen to label the vertices of the triangular pyramid?

SOLUTION

Say the numbers on the four vertices are a, b, c and d.

Hudson has chosen numbers such that $a + b + c$, $a + b + d$, $a + c + d$ and $b + c + d$ are all multiples of 3.

Say we have three numbers, x, y and z, whose sum is a multiple of 3, and let the remainders when x, y and z are divided by 3 be p, q and r [note that p, q and r can only be 0, 1 or 2].

Then $p+q+r$ must also be a multiple of 3 (otherwise there would be a remainder when $x+y+z$ was divided by 3).

So either:

- $p + q + r = 0$, in which case $p = q = r = 0$; or
- $p + q + r = 3$, in which case $p = q = r = 1$ or p, q and r take the values 0, 1 and 2 in some order; or
- $p + q + r = 6$, in which case $p = q = r = 2$.

Hence either all three of x, y and z have the same remainder when divided by 3, or they all have a different remainder when divided by 3.

Going back to a, b, c and d:

if a, b and c all have a different remainder when divided by 3, we can say without loss of generality that a is divisible by 3, b has a remainder of 1 when divided by 3, and c has a remainder of 2.

Then, since $b + c + d$ is divisible by 3, d must be divisible by 3 (since $b + c$ is divisible by 3).

However, then $a + b + d$ would have a remainder of 1 when divided by 3, which isn't allowed, so a, b and c cannot all have a different remainder when divided by 3.

Hence a, b and c must have the same remainder when divided by 3. Once this is established, it quickly becomes clear that d must also have the same remainder when divided by 3.

If that remainder is 1, then a, b, c and d are four of the five numbers 1, 4, 7, 10 and 13. There are 5 ways to choose four numbers from this set (since there are five ways to choose which number to leave out).

If that remainder is 2, then again there Hudson chose four of the five numbers 2, 5, 8, 11 and 14, which gives another five ways to choose the numbers.

And finally, if that remainder is 0, then Hudson chose four of the five numbers 3, 6, 9, 12 and 15, which gives another five ways.

Hence the total number of different sets of numbers could have chosen is $5 + 5 + 5 = 15$.

REMARK

If you have seen *modular arithmetic* before, you will find it a very helpful way to notate an answer which uses this argument.

3. It is possible to write $15129 = 123^2$ as the sum of three distinct squares: $15129 = 27^2 + 72^2 + 96^2$.

(i) By using the identity $(a + b)^2 \equiv a^2 + 2ab + b^2$, or otherwise, find another way to write 15129 as the sum of three distinct squares.

(ii) Hence, or otherwise, show that $378225 = 615^2$ can be written as the sum of six distinct squares.

SOLUTION

(i) Writing 123 as $121 + 2$, we can see that

$$15129 = 123^2 = (121 + 2)^2 = 121^2 + 2(121)(2) + 2^2.$$

Now,

$$2 \times 121 \times 2 = 2 \times 2 \times 121 = 2^2 \times 11^2 = 22^2.$$

So

$$15129 = 121^2 + 22^2 + 2^2.$$

COMMENT

The above solution would earn all the marks, but doesn't really help see how one might discover it. So what might the thought process that leads to this solution look like?

Prompted by the given identity, we might realise it would suffice to find two positive integers a and b such that $a + b = 123$ and $2ab$ is a square.

Trying a few numbers, you might stumble upon 121 and 2 by chance (one of them is very small which makes that more likely), or perhaps you might realise that if you can find a and b with a a square and b twice a square then that would work, since $2ab$ would be $2^2 \times$ square $\times$ square, which is itself square. Choosing $b = 2 \times 1$ (the smallest square) finds you the answer.

There is, in fact, a second solutions you could find using this approach: $98^2 + 70^2 + 25^2$ (when $a = 25$ and $b = 2 \times 49$). .

REMARK

There are in fact 23 ways (ignoring reordering) to write 123 as the sum of the squares of three distinct positive integers.

(2, 22, 121)	(2, 55, 110)	(7, 14, 122)	(7, 34, 118)	(7, 62, 106)
(7, 74, 98)	(12, 36, 117)	(12, 72, 99)	(14, 73, 98)	(22, 71, 98)
(22, 82, 89)	(23, 26, 118)	(23, 50, 110)	(23, 58, 106)	(25, 70, 98)
(26, 62, 103)	(27, 72, 96)	(34, 58, 103)	(36, 72, 93)	(41, 62, 98)
(50, 55, 98)	(58, 62, 89)	(58, 71, 82)		

(ii) Note that $615 = 5 \times 123$. Hence $615^2 = 5^2 \times 123^2$. Since $5^2 = 3^2 + 4^2$, we have

$$\begin{aligned}
615^2 &= (3^2 + 4^2) \times 123^2 \\
&= 3^2 \times 123^2 + 4^2 \times 123^2 \\
&= 3^2 \times [2^2 + 22^2 + 121^2] + 4^2 \times [2^2 + 22^2 + 121^2] \\
&= (3 \times 2)^2 + (3 \times 22)^2 + (3 \times 121)^2 + (4 \times 2)^2 + (4 \times 22)^2 + (4 \times 121)^2 \\
&= 6^2 + 66^2 + 363^2 + 8^2 + 88^2 + 484^2.
\end{aligned}$$

4. Mr Evans has a class containing an even number of students. He calculated that in the end-of-term examination the mean mark of the students was 58, the median mark was 80 and the difference between the lowest mark and the highest mark was 40. Show that Mr Evans made a mistake in his calculations.

Solution

Suppose the class contains $2N$ students, where N is a positive integer, and say the marks of Mr Evans' students were $a_1, a_2, \ldots, a_{2N}$, with $a_1 \leq a_2 \leq \ldots \leq a_{2N}$.

Then we are given

$$a_1 + a_2 + \ldots + a_{2N} = 2N \times 58 = 116N \qquad (1)$$

$$\frac{a_N + a_{N+1}}{2} = 80 \qquad (2)$$

$$a_{2N} - a_1 = 40 \qquad (3)$$

From (2), we know that $a_{N+1} \geq 80$, and hence every $a_j \geq 80$ for $j \geq N + 1$.

Since $a_{2N} \geq 80$, we know that, from (3), $a_1 \geq 40$, and so every $a_i \geq 40$. Then

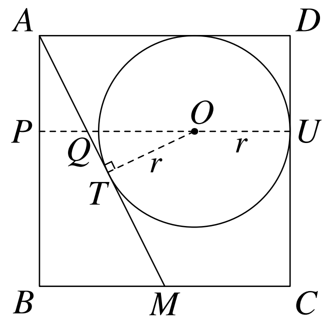
$$\begin{aligned}
&a_1 + a_2 + \ldots + a_{N-1} + a_N + a_{N+1} + a_{N+2} \ldots + a_{2N} \\
&= a_1 + a_2 + \cdots + a_{N-1} + 160 + a_{N+2} + a_{N+3} + \cdots + a_{2N} \\
&\geq \underbrace{40 + 40 + \ldots + 40}_{(N-1)\text{ 40s}} \quad + 160 + \underbrace{80 + 80 + \ldots + 80}_{(N-1)\text{ 80s}} \\
&= 40(N-1) + 160 + 80(N-1) = 120N + 40
\end{aligned}$$

But, comparing with (1), we find this is impossible, since $a_1+a_2+\ldots+a_{2N} = 116N < 120N+40$, so Mr Evans cannot have got his calculations right.

5. A square $ABCD$ has side-length 2, and M is the midpoint of BC. The circle S inside the quadrilateral $AMCD$ touches the three sides AM, CD and DA. What is its radius?

SOLUTION

In any geometry question, drawing a diagram is an important first task:



Let O be the centre of circle S.

Construct the perpendicular to AB through O, and denote the points where it intersects AB, AM and CD as P, Q and U respectively. Let T denote the point where AM meets the circle, and let r denote the radius of the circle.

Note that $AP = r$, and also $AM = \sqrt{5}$ by Pythagoras' Theorem.

Then $\angle OTQ = 90°$ (tangent and radius),

$\angle OQT = \angle BMQ$ (alternate angles).

Hence $\triangle OQT$ is similar to $\triangle AMB$. Hence $\frac{OQ}{OT} = \frac{AM}{AB} = \frac{\sqrt{5}}{2}$, so $OQ = \frac{\sqrt{5}}{2}r$.

Also, $\triangle AQP$ is similar to $\triangle AMB$ (since PQ is parallel to BM), so $\frac{PQ}{AP} = \frac{BM}{AB} = \frac{1}{2}$, so $PQ = \frac{1}{2}AP = \frac{r}{2}$.

Then

$$2 = PQ + QO + OU = \frac{1}{2}r + \frac{\sqrt{5}}{2}r + r = \frac{3+\sqrt{5}}{2}r$$

So

$$r = \frac{2}{\frac{3+\sqrt{5}}{2}} = \frac{4}{3+\sqrt{5}} = 3 - \sqrt{5}.$$

6. A robot sits at zero on a number line. Each second the robot chooses a direction, left or right, and at the sth second the robot moves 2^{s-1} units in that direction on the number line.

For which integers n are there infinitely many routes the robot can take to reach n?

(You may use the fact that every positive integer can be written as a sum of different powers of 2. For example, $19 = 2^0 + 2^1 + 2^4$*)*

SOLUTION

We can denote any route by writing it as the sum of signed powers of two. Then the number to which the route leads is equal to the sum. For example, the route L, R, R, which is a route to 5, can be represented by $-1 + 2 + 4$, which is equal to 5.

First, we note that, after 1 second, the robot is sitting on an odd number (either -1 or $+1$). Every subsequent move will move the robot an even number of units along, so the robot cannot reach any even numbers.

We now claim that it is possible for the robot to reach any positive odd number.

Let our positive odd number be m. Using the fact given at the end of the problem, we can write m as a sum of different powers of 2. Write these powers of 2 in increasing order.

For example, when $m = 7$, we have

$$7 = 2^0 + 2^1 + 2^2 = 1 + 2 + 4$$

The largest power of 2 that appears is 4, and every smaller power of 2 also appears.

When $m = 89$, we have

$$89 = 2^0 + 2^3 + 2^4 + 2^6 = 1 + 8 + 16 + 64$$

The largest power of 2 here is 64, but some smaller powers of 2 are missing $(2, 4$ and $32)$.

Now, if every power of 2 up to the largest power of 2 appears in the sum (like for $m = 7$), we immediately have a route to m — the robot always moves to the right until they reach m.

If, however, there is a gap between a pair of powers of 2 that appear in the sum (there are two such gaps when $m = 89$), then we can fill in the gap as follows:

- If the sum contains a term 2^a but not 2^{a+1}, replace 2^a with $-2^a + 2^{a+1}$.
- Repeat until the gap is filled.

The sum remains the same throughout the process, because $-2^a + 2^{a+1} = -2^a + 2(2^a) = 2^a$

For example, when $m = 89$, we fill in the first gap (between 1 and 8) like this:

$$\begin{aligned} 89 &= 1 + 8 + 16 + 64 \\ &= -1 + 2 + 8 + 16 + 64 \\ &= -1 - 2 + 4 + 8 + 16 + 64 \end{aligned}$$

and fill in the second gap (between 16 and 64) like this:

$$\begin{aligned}89 &= -1 - 2 + 4 + 8 + 16 + 64\\ &= -1 - 2 + 4 + 8 - 16 + 32 + 64\end{aligned}$$

We now have a route to 89: L, L, R, R, L, R, R.

It follows that there is a route to every positive odd number m. In particular, we can find a route that ends with an R.

This allows us to find a new route m, by replacing the final R with L, R. The sum will remain the same for the same reason as in the gap-filling process above.

So given a route to a positive odd number m, we can always build a new (longer) route to m. Hence there are infinitely many routes to m.

Finally, any route to a positive odd number m can be converted into a route to the negative odd number $-m$, by changing the direction of each step on the route (replace all Ls with Rs and vice versa).

Hence, the numbers for which there are infinitely many routes the robot can take are precisely the odd numbers (positive or negative).